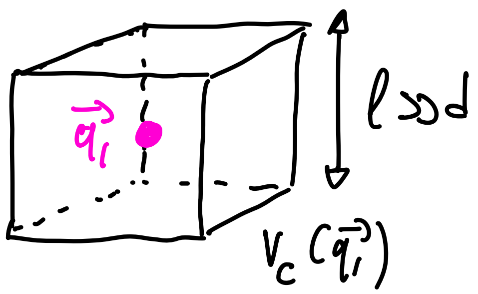
$$\partial_t f_1 + \{ f_1 , H_1 \} = \int d\vec{q}_2 \, d\vec{p}_2 \, \frac{\partial V(\vec{q}_1 - \vec{q}_2)}{\partial \vec{q}_1} \cdot \frac{\partial}{\partial \vec{p}_1} f_2(\vec{q}_1, \vec{p}_1, \vec{q}_2, \vec{p}_2) \qquad (F_1)$$

Start from (F1) & build a coarse grained description over scales τ, ℓ such that

$$\tau_{MFP} \gg \tau \gg \tau_{col} \quad \& \quad \ell_{MFP} \gg \ell \gg d$$



$$\hat{f}_1(\vec{q}_1, \vec{p}_1, t) = \frac{1}{V_c} \int_{V_c(\vec{q}_1)} d^3\vec{q} \, f_1(\vec{q}, \vec{p}_1, t)$$

$$\partial_t f_1 + \{ f_1 , H_1 \} \xrightarrow[\text{space}]{\frac{1}{V_c}\int d^3\vec{q}_1} \partial_t \hat{f}_1 + \{ \hat{f}_1 , H_1 \}$$

$H_1 \sim$ constant over V_c

time

$\int_t^{t+d\tau} ds$

$f_1 \sim$ constant over τ

$$\tau \left[\partial_t \hat{f}_1 + \{ \hat{f}_1 , H_1 \} \right]$$

What about collisions?

Collisions On a time $\tau_{col} << \tau << \tau_{MFP}$, there are very few collisions, which appear as rare & random encounters between particles $\Rightarrow$ build stochastic description

Let us denote $d\Gamma(\vec{q}_1, \vec{p}_1)$ the phase space volume

$$d\Gamma(\vec{q}_1, \vec{p}_1) \equiv V_c(\vec{q}_1) \times \prod_{\alpha=x,y,z} [p_{1,\alpha} ; p_{1,\alpha} + dp_{1,\alpha}] .$$

During τ, in $d\Gamma$, collisions $\vec{p}_1\,'' , \vec{p}_2\,'' \xrightarrow{(1)} \vec{p}_1\,' , \vec{p}_2\,'$ increase $f_1(\vec{q}_1\,', \vec{p}_1\,', t)$ while collisions $\vec{p}_1\,', \vec{p}_2 \xrightarrow{(2)} \vec{p}_1\,'' , \vec{p}_2\,''$ decrease it.

$$\underbrace{d\Gamma}_{\text{to go from density to numbers}} \int_t^{t+\tau} ds\, \partial_t \hat{f}_1 \Big|_{col} = \text{Variations of average number of particles in } d\Gamma(\vec{q}_1, \vec{p}_1) \text{ over } \tau$$

$$\equiv N^+ - N^-$$

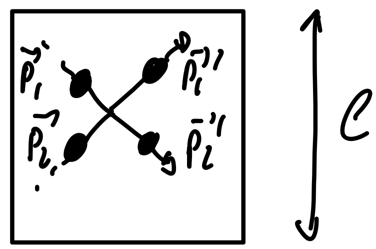
where N^+ = average number entering $d\Gamma$ due to a type (1) collision &

N^- = ———— leaving ———— type (2) ——.

$\Rightarrow$ Need to compute N^+ & N^-, then $\displaystyle\int_t^{t+\tau} ds\, \partial_t \hat{f}_1\Big|_{col} \simeq \frac{N^+ - N^-}{d\Gamma}$

dilute gas: Since the gas is dilute, we neglect three-body interactions and thus only consider processes $\vec{q}_1, \vec{p}_1, \vec{q}_2, \vec{p}_2 \to \vec{q}_1{}', \vec{p}_1{}', \vec{q}_2{}', \vec{p}_2{}'$

Locality: Since $l \gg d$, collisions happen within a single volume $V_l \Rightarrow \vec{q}_1, \vec{q}_2, \vec{q}_1{}', \vec{q}_2{}' = \vec{q}_1$



Molecular chaos approximation: the average number of pairs in the box with momenta $\vec{p}_1$ & $\vec{p}_2$ is proportional to $\hat{\rho}_2(\vec{q}_1, \vec{p}_1, \vec{q}_1, \vec{p}_2) \Rightarrow$ joint probability density for particle 1 & 2.

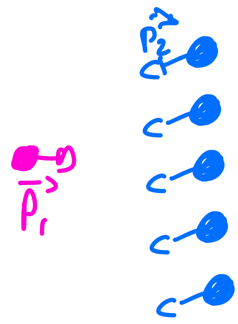
Since the system is dilute, it is unlikely that particle 1 & 2 have collided recently $\Rightarrow$ assumed independent

$$\rho_2(\vec{q}_1, \vec{p}_1, \vec{q}_1, \vec{p}_2) \simeq \rho_1(\vec{q}_1, \vec{p}_1)\, \rho_1(\vec{q}_1, \vec{p}_2)$$

$$\Rightarrow \hat{f}_2(\vec{q}_1, \vec{p}_1, \vec{q}_2, \vec{p}_2) \simeq \hat{f}_1(\vec{q}_1, \vec{p}_1)\, \hat{f}_1(\vec{q}_1, \vec{p}_2)$$

Loss term N^-: $\vec{p}_1, \vec{p}_2 \to \vec{p}_1', \vec{p}_2'$

$$N^- = \underbrace{\hat{f}_1(\vec{q}_1, \vec{p}_1, t)\, V_c\, d^3\vec{p}_1}_{\#\text{ of part with } \vec{q}_1, \vec{p}_1} \times \underbrace{\tau}_{\text{duration}} \times \underbrace{R^-(\vec{p}_1)}_{\text{rate of collision with other particles}}$$



$R^-(\vec{p}_1) \to$ sum over all possible $\vec{p}_2$

In the frame of reference of particle 1, particle 2 has velocity $\vec{v}_2 - \vec{v}_1$.

$$\Rightarrow \underbrace{R^-(\vec{p}_1)}_{[T]^{-1}} = \underbrace{\int d^3\vec{p}_2\, |\vec{v}_2 - \vec{v}_1|\, \hat{f}_1(\vec{q}_1, \vec{p}_2, t)}_{[L^{-2} \cdot T^{-1}]} \times \underbrace{(\text{something})}_{[L^2]}$$

"(something)" is thus an area that tells us what fraction of incoming particles with $\vec{p}_2$ will collide with particle 1.

$\Rightarrow$ Cross section $\sigma(\vec{p}_2, \vec{p}_1)$

E.g., for hard spheres $\sigma = \pi d^2$.
In the presence of attractive interactions, $\sigma > \pi d^2$.

$$\boxed{N^- = \hat{f}_1(\vec{q}_1, \vec{p}_1, t)\, V_c\, d^3\vec{p}_1\, \tau \int d^3\vec{p}_2\, |\vec{v}_1 - \vec{v}_2|\, \hat{f}_1(\vec{q}_1, \vec{p}_2, t)\, \sigma(\vec{p}_1, \vec{p}_2)}$$

We do not need to know σ explicitly to prove convergence to equilibrium, but we need to know some of its properties.

Note that a collision between $\vec{p}_1$ & $\vec{p}_2$ may lead to a variety of $\vec{p}_1{}'$ & $\vec{p}_2{}'$

$$\sigma(\vec{p}_1,\vec{p}_2) = \int_{\substack{\text{admissible}\\ \vec{p}_1{}' \,\&\, \vec{p}_2{}'}} d\sigma(\vec{p}_1,\vec{p}_2 \to \vec{p}_1{}',\vec{p}_2{}')$$

Q: How can we characterize the admissible values & compare $d\sigma(\vec{p}_1,\vec{p}_2 \to \vec{p}_1{}',\vec{p}_2{}')$ with $d\sigma(\vec{p}_1{}',\vec{p}_2{}' \to \vec{p}_1,\vec{p}_2)$?

Conservation of momentum & energy

$\vec{p}_1+\vec{p}_2 = 2\vec{p}_{CM} = \vec{p}_1{}'+\vec{p}_2{}' = 2\vec{p}_{CM}{}'$ $\Rightarrow$ The momentum of the center of mass is a collisional invariant

$$2mE = \vec{p}_1{}^2+\vec{p}_2{}^2 = \frac{1}{2}\left[(\vec{p}_1+\vec{p}_2)^2+(\vec{p}_1-\vec{p}_2)^2\right] = 2\vec{p}_{CM}{}^2 + 2\vec{p}_d{}^2 \ ; \quad \vec{p}_d = \frac{\vec{p}_1-\vec{p}_2}{2}$$

Since E & $\vec{p}_{CM}{}^2$ are invariant, so is $\vec{p}_d{}^2$. $\vec{p}_d$ is called the relative momentum. $\|\vec{p}_d{}'\| = \|\vec{p}_d\| \Rightarrow$ live on the same sphere $\Rightarrow$ parametrize using

$\vec{p}_d{}' = \mathrm{Rot}(\theta,\varphi)\,\vec{p}_d \Rightarrow$ all possible outgoing momenta are parametrized by the (Euler angles of the) rotation that maps $\vec{p}_d$ at $\vec{p}_d{}'$.

$$d^2\sigma(\vec{p}_1,\vec{p}_2 \to \vec{p}_1{}',\vec{p}_2{}') = d^2\sigma(\theta,\varphi) \quad \& \quad \int_{\substack{\text{admissible}\\ \text{values}}} = \int_{\theta,\varphi}$$

$$\boxed{N^- = \hat{f}_1(\vec{q}_1,\vec{p}_1,t)\, V_c\, d^3\vec{p}_1\, \tau \int d^3\vec{p}_2\, \underbrace{d^2\sigma(\theta,\varphi)}_{\equiv\, d^2\sigma(\vec{p}_1,\vec{p}_2 \to \vec{p}_1{}',\vec{p}_2{}')}\, |\vec{v}_1-\vec{v}_2|\, \hat{f}_1(\vec{q}_1,\vec{p}_2,t)}$$

Gain term N^+:

We now need to consider collisions that lead to $\vec{p}_1, \vec{p}_2$, from $\vec{p}_1{}', \vec{p}_2{}'$.

We first note that

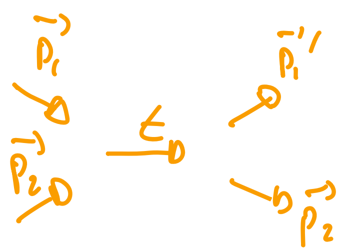
$$N^- = \int dN^- = \int_{\vec{p}_2, \theta, \varphi} dN(\vec{p}_2, \vec{p}_1 \to \vec{p}_2{}', \vec{p}_1{}')$$

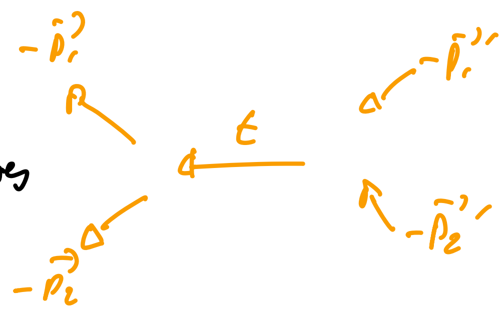
Similarly we can define

$$dN^+ = dN(\vec{p}_2{}', \vec{p}_1{}' \to \vec{p}_2, \vec{p}_1)$$
$$= V_c \tau \, d^3\vec{p}_1{}' \, d^3\vec{p}_2{}' \, d^2\sigma(\vec{p}_1{}', \vec{p}_2{}' \to \vec{p}_1, \vec{p}_2) \, \hat{f}_1(\vec{p}_1{}', \vec{q}_1) \, \hat{f}_1(\vec{p}_2{}', \vec{q}_1) \, |\vec{v}_1{}' - \vec{v}_2{}'| \quad (1)$$

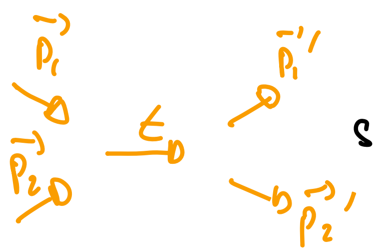
Can we put this in a form where it is more easily compared with dN^-?

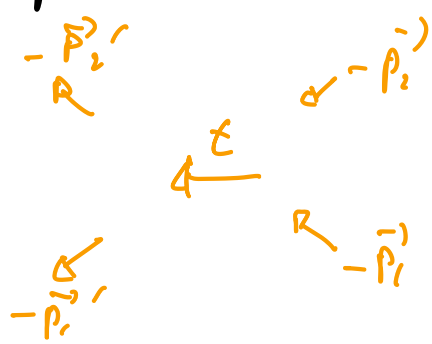
* Time reversal symmetry

If 

solves Hamilton's eq°, so does 

* Parity symmetry (rotation by π in the $(\vec{p}_1, \vec{p}_2)$ plane)

If 

solves Hamilton's eq°, so does 

$$\Rightarrow d^2\sigma(\vec{p}_1, \vec{p}_2 \to \vec{p}_1{}', \vec{p}_2{}') \overset{TRS}{=} d^2\sigma(-\vec{p}_1{}', -\vec{p}_2{}' \to -\vec{p}_1, -\vec{p}_2) \overset{PS}{=} d^2\sigma(\vec{p}_1{}', \vec{p}_2{}' \to \vec{p}_1, \vec{p}_2) \quad (2)$$

Backward & forward scattering have the same efficiency.

* Collision = rotation of $\vec{p}_d = \frac{\vec{p}_1 - \vec{p}_2}{2}$

① $|\vec{v}_1{}' - \vec{v}_2{}'| = |\vec{v}_1 - \vec{v}_2|$ (3)

(ii) $\vec{p}_1 = \vec{p}_d + \vec{p}_{CM}$; $\vec{p}_2 = \vec{p}_{CM} - \vec{p}_d$; $k = \begin{vmatrix} \frac{d\vec{p}_1}{d\vec{p}_{CM}} & \frac{d\vec{p}_1}{d\vec{p}_d} \\ \frac{d\vec{p}_2}{d\vec{p}_{CM}} & \frac{d\vec{p}_2}{d\vec{p}_d} \end{vmatrix}$

$$d\vec{p}_1\, d\vec{p}_2 = k\, d\vec{p}_{CM}\, d\vec{p}_d = k\, d\vec{p}_{CM}{}'\, d\vec{p}_d{}' = d\vec{p}_1{}'\, d\vec{p}_2{}' \quad (4)$$

$\vec{p}_{CM} = \vec{p}_{CM}{}'$

$\vec{p}_d = \mathrm{Rot}(\theta, \varphi) . \vec{p}_d{}'$

Explicit computation:

$$\vec{p}_1{}' = \frac{1}{2}(\vec{p}_1 + \vec{p}_2) + \mathrm{Rot} . \frac{1}{2}(\vec{p}_1 - \vec{p}_2) = \frac{Id + Rot}{2}\vec{p}_1 + \frac{Id - Rot}{2}\vec{p}_2$$

$$\vec{p}_2{}' = \frac{1}{2}(\vec{p}_1 + \vec{p}_2) - \mathrm{Rot} . \frac{1}{2}(\vec{p}_1 - \vec{p}_2) = \frac{Id - Rot}{2}\vec{p}_1 + \frac{Id + Rot}{2}\vec{p}_2$$

$$\text{Jacobian: } 2^{-6} \begin{array}{c} \begin{vmatrix} Id + Rot & Id - Rot \\ Id - Rot & Id + Rot \end{vmatrix} \\ \begin{array}{cc} C_1 & C_2 \end{array} \end{array} = 2^{-6} \begin{array}{cc} \begin{vmatrix} Id + Rot & 2Id \\ Id - Rot & 2Id \end{vmatrix} & \begin{array}{c} L_1 \\ L_2 \end{array} \\ \begin{array}{cc} C_1 & C_1 + C_2 \end{array} & \end{array}$$

$$= 2^{-6} \begin{array}{cc} \begin{vmatrix} 2Rot & 0 \\ Id - Rot & 2Id \end{vmatrix} & \begin{array}{c} L_1 - L_2 \\ L_2 \end{array} \end{array} = 2^{-6} . 2^{6} = 1$$

Injecting (2), (3), (4) into (1) leads to

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$$\Rightarrow dN^+ = V_c \tau \, d^3\vec{p}_1' \, d^3\vec{p}_2' \quad d^2\sigma(\vec{p}_1', \vec{p}_2' \to \vec{p}_1, \vec{p}_2) \, \hat{f}_1(\vec{p}_1', \vec{q}_1) \hat{f}_1(\vec{p}_2', \vec{q}_1) |\vec{v}_1 - \vec{v}_2|$$

$$\& \; N^+ = V_c \tau \, d^3\vec{p}_1 \int d^3\vec{p}_2 \, d^2\sigma(\vec{p}_1, \vec{p}_2 \to \vec{p}_1', \vec{p}_2') \, \hat{f}_1(\vec{p}_1', \vec{q}_1) \hat{f}_1(\vec{p}_2', \vec{q}_1) |\vec{v}_1 - \vec{v}_2|$$

$$\text{Thus} \int_t^{t+\tau} ds \left.\frac{\partial \hat{f}_1}{\partial t}\right|_{col} = \frac{N^+ - N^-}{V_c d^3\vec{p}_1} = \tau \int d^3\vec{p}_2 \, d^2\sigma(\vec{p}_1, \vec{p}_2 \to \vec{p}_1', \vec{p}_2') |\vec{v}_1 - \vec{v}_2| \Big[\hat{f}_1(\vec{p}_1', \vec{q}_1) \hat{f}_1(\vec{p}_1', \vec{q}_2) - \hat{f}_1(\vec{p}_1, \vec{q}_1) \hat{f}_1(\vec{p}_2, \vec{q}_2) \Big]$$

All in all, this leads to the celebrated Boltzmann equation

$$\boxed{\frac{\partial \hat{f}_1(\vec{q}_1, \vec{p}_1, t)}{\partial t} + \{\hat{f}_1, H_1\} = \int d^3\vec{p}_2 \, d^2\sigma \, |\vec{v}_1 - \vec{v}_2| \Big[\hat{f}_1(\vec{p}_1', \vec{q}_1) \hat{f}_1(\vec{p}_1', \vec{q}_2) - \hat{f}_1(\vec{q}_1, \vec{p}_1) \hat{f}_1(\vec{q}_1, \vec{p}_2) \Big]} \quad \text{(BE)}$$

which is a closed equation for $\hat{f}_1$!